

# EFFECTIVITY OF BRAUER–MANIN OBSTRUCTIONS ON SURFACES

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ABSTRACT. We study Brauer–Manin obstructions to the Hasse principle and to weak approximation on algebraic surfaces over number fields.

## 1. INTRODUCTION

Let  $X$  be a smooth projective variety over a number field  $k$ . An important area of research concerns the behavior of the set of  $k$ -rational points  $X(k)$ . One of the major open problems is the decidability problem for  $X(k) \neq \emptyset$ . An obvious necessary condition is the existence of points over all completions  $k_v$  of  $k$ ; this can be effectively tested given defining equations of  $X$ . One says that  $X$  *satisfies the Hasse principle* when

$$X(k) \neq \emptyset \Leftrightarrow X(k_v) \neq \emptyset \quad \forall v. \quad (1.1)$$

One well-studied obstruction to this is the *Brauer–Manin obstruction* [Man71]. It has proved remarkably useful in explaining counterexamples to the Hasse principle, especially on curves [Sto07] and geometrically rational surfaces [CSS87]; see also [Sko01]. Although there are counterexamples not explained by the Brauer–Manin obstruction [Sko99], [Po10], there remains a wide class of algebraic varieties for which the sufficiency of the Brauer–Manin obstruction is a subject of active research. This includes  $K3$  surfaces, studied for instance in [Swi00], [Wit04], [HS05], [SS05], [Bri06], [Ie], [HVV].

We recall, that an element  $\alpha \in \text{Br}(X)$  cuts out a subspace

$$X(\mathbb{A}_k)^\alpha \subseteq X(\mathbb{A}_k)$$

of the adelic space, defined as the set of all  $(x_v) \in X(\mathbb{A}_k)$  satisfying

$$\sum_v \text{inv}_v(\alpha(x_v)) = 0.$$

Here,  $\text{inv}_v$  is the local invariant of the restriction of  $\alpha$  to a  $k_v$ -point, taking its value in  $\mathbb{Q}/\mathbb{Z}$ . By the exact sequence of class field theory

$$0 \rightarrow \text{Br}(k) \rightarrow \bigoplus_v \text{Br}(k_v) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

(here  $\text{inv}$  is the sum of  $\text{inv}_v$ ), we have

$$X(k) \subseteq X(\mathbb{A}_k)^\alpha.$$

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Therefore, for any subset  $B \subseteq \mathrm{Br}(X)$  we have

$$X(k) \subseteq X(\mathbb{A}_k)^B := \bigcap_{\alpha \in B} X(\mathbb{A}_k)^\alpha.$$

A natural goal is to be able to compute the space  $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$  *effectively*. By this we mean, to give an algorithm, for which there is an *a priori* bound on the running time, in terms of the input data (e.g., the defining equations of  $X$ ). The existence of such an effective algorithm was proved for geometrically rational surfaces in [KT08]. Here we prove the following result.

**Theorem 1.** *Let  $X$  be a smooth projective geometrically irreducible surface over a number field  $k$ , given by a system of homogeneous polynomial equations. Assume that the geometric Picard group  $\mathrm{Pic}(X_{\bar{k}})$  is torsion free and generated by finitely many divisors, each with a given set of defining equations. Then for each positive integer  $n$  there exists an effective description of a space  $X_n \subseteq X(\mathbb{A}_k)$  which satisfies*

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)} \subseteq X_n \subseteq X(\mathbb{A}_k)^{\mathrm{Br}(X)[n]},$$

where  $\mathrm{Br}(X)[n] \subseteq \mathrm{Br}(X)$  denotes the  $n$ -torsion subgroup. In particular,  $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$  is effectively computable provided that  $|\mathrm{Br}(X)/\mathrm{Br}(k)|$  can be bounded effectively.

For instance, in the case of a diagonal quartic surface over  $\mathbb{Q}$  there is an effective bound on  $|\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})|$  due to Ieronymou, Skorobogatov, and Zarhin [ISZ].

While it is not known how to compute  $\mathrm{Pic}(X_{\bar{k}})$  effectively, in general, there is a method of computation involving reduction modulo primes used by van Luijk [vL07]; further examples can be found in [EJ08] and [HVV].

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## 2. PICARD SCHEMES

Let  $X \rightarrow S$  be a finite-type morphism of locally Noetherian schemes. We recall that the functor associating to an  $S$ -scheme  $T$  the group

$$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X \times_S T) / \mathrm{Pic}(T)$$

is known as the *relative Picard functor*. It restricts to a sheaf on the étale site  $S_{\mathrm{et}}$  when  $S$  is a nonsingular curve over an algebraically closed field, by Tsen's theorem. See [Kle05].

We use  $\mathrm{Br}(X)$  to denote the cohomological Brauer group of a Noetherian scheme  $X$ , i.e., the torsion subgroup of the étale cohomology group  $H^2(X, \mathbb{G}_m)$ . When  $X$  is regular,  $H^2(X, \mathbb{G}_m)$  is itself a torsion group. By Gabber's theorem, if  $X$  admits an ample invertible sheaf then  $\mathrm{Br}(X)$  is also equal to the Azumaya Brauer group, i.e., the equivalence classes of sheaves of Azumaya algebras on  $X$ . For background on the Brauer group, the reader is referred to [Gro68], and for a proof of Gabber's theorem, see [dJ05].

Let  $S$  be a nonsingular irreducible curve over an algebraically closed field, and let  $f : X \rightarrow S$  be a smooth projective morphism of relative dimension 1 with connected fibers. Then the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q f_* \mathbb{G}_m) \implies H^{p+q}(X, \mathbb{G}_m)$$

gives, by [Gro68, Cor. III.3.2], an isomorphism

$$\mathrm{Br}(X) \xrightarrow{\sim} H^1(S, \mathrm{Pic}_{X/S}). \quad (2.1)$$

Furthermore, we have an exact sequence

$$0 \rightarrow \mathrm{Pic}_{X/S}^0 \rightarrow \mathrm{Pic}_{X/S} \rightarrow \mathbb{Z} \rightarrow 0$$

of sheaves (on  $S_{\mathrm{et}}$ ) hence an exact sequence

$$0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow H^1(S, \mathrm{Pic}_{X/S}^0) \rightarrow H^1(S, \mathrm{Pic}_{X/S}) \rightarrow 0, \quad (2.2)$$

where  $d$  is the gcd of the relative degrees of all multisections of  $f$ . Now assume that the algebraically closed base field has characteristic not dividing  $n$ . Then we have the exact sequence of sheaves

$$0 \rightarrow \mathrm{Pic}_{X/S}[n] \rightarrow \mathrm{Pic}_{X/S}^0 \xrightarrow{n} \mathrm{Pic}_{X/S}^0 \rightarrow 0$$

(exactness on the right follows by [EGAIV, 21.9.12] and [Kle05, Prop. 9.5.19]) from which the long exact sequence in cohomology gives a surjective homomorphism

$$H^1(S, \mathrm{Pic}_{X/S}[n]) \rightarrow H^1(S, \mathrm{Pic}_{X/S}^0)[n]. \quad (2.3)$$

**Lemma 2.** *Let  $K$  be a field, and let  $D$  be a geometrically irreducible smooth projective curve over  $K$ . Let  $n$  be a positive integer, not divisible by  $\mathrm{char}(K)$ . Let  $C$  be a nonempty open subset of  $D$ , with  $Y := D \setminus C$  nonempty. The inclusions will be denoted  $i: Y \rightarrow D$  and  $j: C \rightarrow D$ .*

- (i) *We have  $R^1 j_* \mu_n = i_*(\mathbb{Z}/n\mathbb{Z})$ .*
- (ii) *For a tuple of integers  $(a_y)_{y \in Y}$  with reductions  $(\bar{a}_y)$  modulo  $n$ , we have  $(\bar{a}_y)$  in the image of the map*

$$H^1(C, \mu_n) \rightarrow H^0(D, R^1 j_* \mu_n) = \bigoplus_{y \in Y} \mathbb{Z}/n\mathbb{Z}$$

*coming from the Leray spectral sequence if and only if there exists a divisor  $\delta$  on  $D$  with  $n\delta \sim \sum a_y [y]$ , where  $\sim$  denotes linear equivalence of divisors.*

*Proof.* The Leray spectral sequence gives

$$0 \rightarrow H^1(D, \mu_n) \rightarrow H^1(C, \mu_n) \rightarrow H^0(D, R^1 j_* \mu_n) \xrightarrow{d_2^{0,1}} H^2(D, \mu_n) \rightarrow H^2(C, \mu_n). \quad (2.4)$$

For (i), by standard spectral sequences we have  $R^1 j_* \mu_n = i_* \underline{H}_Y^2(\mu_n)$  (cf. [Mil80, proof of Thm. VI.5.1]). So we are reduced to a local computation, and we may therefore assume that  $D$  is affine and  $Y$  consists of a single point which is a principal Cartier divisor on  $D$ . By the Kummer sequence and injectivity of  $\mathrm{Br}(C) \rightarrow \mathrm{Br}(D)$  the right-hand map in (2.4) is injective, while the left-hand map has cokernel cyclic of order  $n$ . (Such an isomorphism exists generally for regular codimension 1 complements, see [SGA4, (XIX.3.3)].)

For the “if” direction of (ii), we take  $r \in K(D)^*$  to be a rational function whose divisor is  $-n\delta + \sum a_y [y]$ . Then adjoining  $r^{1/n}$  to the function field of  $D$  yields an element of  $H^1(C, \mu_n)$  whose image in  $H^0(D, R^1 j_* \mu_n)$  is  $(\bar{a}_y)$  by the isomorphism in (i). For the “only if” direction, an element of  $H^1(C, \mu_n)$  gives rise by the Kummer exact sequence to a divisor  $\delta$  on  $C$  and  $r \in K(C)^*$  by which  $n\delta \sim 0$  on  $C$ . Then  $n\delta \sim \sum b_y [y]$  on  $D$ , for some integers  $b_y$ , and the given element of  $H^1(C, \mu_n)$  maps by  $d_2^{0,1}$  to  $(\bar{b}_y)$ . This means that  $a_y \equiv b_y \pmod{n}$  for all  $y \in Y$ , and we easily obtain  $\delta'$  on  $D$  with  $n\delta' \sim \sum a_y [y]$ .  $\square$

## 3. BRAUER GROUPS

We start with some general results about cocycles in étale cohomology.

**Lemma 3.** *Let  $X$  be a Noetherian scheme, union of open subschemes  $X_1$  and  $X_2$ , and let  $G$  be an abelian étale sheaf. Suppose given étale covers  $Y_i \rightarrow X_i$  and Čech cocycles  $\beta_i \in Z^2(Y_i \rightarrow X_i, G)$  for  $i = 1, 2$ . With  $X_{12} = X_1 \cap X_2$  and  $Y_{12} = Y_1 \times_X Y_2$ , we suppose further that a cochain  $\delta \in C^1(Y_{12} \rightarrow X_{12}, G)$  is given, satisfying*

$$\frac{\delta(y_1, y'_1, y_2, y'_2) \delta(y'_1, y''_1, y'_2, y''_2)}{\delta(y_1, y'_1, y_2, y''_2)} = \frac{\beta_1(y_1, y'_1, y''_1)}{\beta_2(y_2, y'_2, y''_2)}$$

for  $(y_1, y_2, y'_1, y'_2, y''_1, y''_2) \in Y_{12} \times_X Y_{12} \times_X Y_{12}$ . Then we have  $\beta \in Z^2(Y_1 \amalg Y_2 \rightarrow X, G)$ , given by

$$\begin{aligned} (y_1, y'_1, y''_1) &\mapsto \beta_1(y_1, y'_1, y''_1), \\ (y_1, y'_1, y'_2) &\mapsto \delta(y_1, y'_1, y'_2, y'_2) \beta_2(y'_2, y'_2, y'_2), \\ (y_1, y'_2, y'_1) &\mapsto \delta(y_1, y'_1, y'_2, y'_2)^{-1} \beta_2(y'_2, y'_2, y'_2)^{-1}, \\ (y_1, y'_2, y'_2) &\mapsto \delta(y_1, y_1, y'_2, y'_2)^{-1} \beta_1(y_1, y_1, y_1), \\ (y_2, y'_1, y''_1) &\mapsto \delta(y'_1, y'_1, y_2, y_2) \beta_2(y_2, y_2, y_2), \\ (y_2, y'_1, y'_2) &\mapsto \delta(y'_1, y'_1, y_2, y'_2) \beta_1(y'_1, y'_1, y'_1)^{-1}, \\ (y_2, y'_2, y'_1) &\mapsto \delta(y'_1, y'_1, y_2, y'_2)^{-1} \beta_1(y'_1, y'_1, y'_1), \\ (y_2, y'_2, y'_2) &\mapsto \beta_2(y_2, y'_2, y'_2), \end{aligned}$$

whose class restricts to the class of  $\beta_i$  in  $H^2(X_i, G)$  for  $i = 1, 2$ .

*Proof.* This is just a portion of the Mayer-Vietoris sequence, written out explicitly in terms of cocycles.  $\square$

The following two results are based on the existence of Zariski local trivializations of 1-cocycle with values in  $\mathbb{G}_m$ . Such trivializations exist effectively when the 1-cocycle is effectively presented, say on a scheme of finite type over a number field.

**Lemma 4.** *Let  $k$  be a number field,  $X$  a finite-type scheme over  $k$ ,  $Y \rightarrow X$  and  $Z \rightarrow X$  finite-type étale covers, and  $Y \rightarrow Z$  a morphism over  $X$ . Suppose that  $\beta \in Z^2(Z \rightarrow X, \mathbb{G}_m)$  and  $\delta \in C^1(Y \rightarrow X, \mathbb{G}_m)$  are given, so that the restriction of  $\beta$  by  $Y \rightarrow Z$  is equal to the coboundary of  $\delta$ . Then we may effectively produce a Zariski open covering  $Z = \bigcup_{i=1}^N Z_i$  for some  $N$  and a 1-cochain for  $\coprod_{i=1}^N Z_i \rightarrow X$  whose coboundary is equal to the restriction of  $\beta$  by  $\coprod_{i=1}^N Z_i \rightarrow Z$ .*

*Proof.* Replacing  $Y$  by  $Y \times_X Z$  and using that the restriction maps on the level of Čech cocycles corresponding to  $Y \times_X Z \rightarrow Y \rightarrow Z$  and  $Y \times_X Z \rightarrow Z$  differ by an explicit coboundary (cf. [Mil80, Lem. III.2.1]), we are reduced to the case that  $Y \rightarrow Z$  is also a covering.

Then we have the 1-cocycle for  $Y \rightarrow Z$

$$(y, y') \mapsto \frac{\beta(z, z, z)}{\delta(y, y')},$$

for  $(y, y') \in Y \times_Z Y$  over  $z$ . This may be trivialized effectively on a Zariski open neighborhood of any point of  $Z$ , so we may effectively obtain a refinement of  $Z$  to a

Zariski open covering and functions  $\varepsilon_i$  satisfying

$$\frac{\varepsilon_i(y')}{\varepsilon_i(y)} = \frac{\beta(z, z, z)}{\delta(y, y')}$$

for all  $i$  and  $(y, y') \in Y \times_Z Y$  over  $z \in Z_i$ . It follows that for all  $i$  and  $j$ , and  $(y, y') \in Y \times_X Y$  over  $(z, z') \in Z \times_X Z$  with  $z \in Z_i, z' \in Z_j$ , the function

$$\frac{\varepsilon_j(y')}{\varepsilon_i(y)} \delta(y, y')$$

depends only on  $(z, z')$ , hence we obtain  $\delta_0 \in C^1(\coprod_{i=1}^N Z_i \rightarrow X)$  satisfying

$$\frac{\varepsilon_j(y')}{\varepsilon_i(y)} \delta(y, y') = \delta_0(z, z').$$

The conclusion follows immediately from this formula.  $\square$

**Lemma 5.** *Let  $X$  be a smooth finite-type scheme over a number field  $k$ , let  $Z \rightarrow X$  be a finite-type étale covering, and let  $Y \rightarrow Z$  be a finite-type étale morphism with dense image. Let  $\beta \in Z^2(Z \rightarrow X, \mathbb{G}_m)$  be given, along with  $\delta \in \mathcal{O}_{Y \times_X Y}^*$  satisfying*

$$\delta(y, y') \delta(y', y'') / \delta(y, y'') = \beta(z, z', z''),$$

*for all  $(y, y', y'') \in Y \times_X Y \times_X Y$  over  $(z, z', z'') \in Z \times_X Z \times_X Z$ . Then there exists, effectively, a Zariski open covering  $(Z_i)_{1 \leq i \leq N}$  of  $Z$  (for some  $N$ ) and a 1-cocycle for  $\coprod Z_i \rightarrow X$  whose coboundary is the restriction of  $\beta$  by  $\coprod Z_i \rightarrow Z$ .*

*Proof.* Let  $X_0$  denote the image of the composite morphism  $Y \rightarrow X$ , and  $Z_0$  the pre-image of  $X_0$  in  $Z$ . By Lemma 4 (or rather its proof) there exists a Zariski open covering of  $Z_0$  of the form  $(Z_0 \cap Z_i)_{1 \leq i \leq N}$  for some Zariski open covering  $(Z_i)$  of  $Z$  (the 1-cocycle mentioned in the proof determines a line bundle on  $Z_0$ , which can be extended to a line bundle on  $Z$ , since  $Z$  is smooth) and a 1-cochain for  $\coprod Z_i \cap Z_0 \rightarrow X_0$  whose coboundary is the restriction of  $\beta$ . Using the fact that divisors on smooth schemes are locally principal (and effectively so, e.g., see [KT08, §7]) and [Mil80, Exa. III.2.22], we see that after further refinement of  $(Z_i)$  the 1-cochain extends to a 1-cochain for  $\coprod Z_i \rightarrow X$ .  $\square$

Let  $X$  be a regular Noetherian scheme of dimension 2. It is known [Gro68, Cor. II.2.2] that for any element  $\alpha \in \text{Br}(X)$  of the (cohomological) Brauer group there exists a sheaf of Azumaya algebras on  $X$  having class equal to  $\alpha$ .

**Lemma 6.** *Let  $X$  be a smooth projective surface over a number field  $k$ ,  $\widehat{X} \subset X$  an open subscheme whose complement has codimension 2, and  $\alpha \in \text{Br}(X)$  an element whose restriction over  $\widehat{X}$  is represented by a 2-cocycle  $\widehat{\beta}$ , relative to some finite-type étale cover  $\pi: \widehat{Y} \rightarrow \widehat{X}$ . We suppose that  $X, \widehat{X}, \widehat{Y}, \pi$ , and  $\widehat{\beta}$  are given by explicit equations. Then there is an effective procedure to produce a sheaf of Azumaya algebras on  $X$  representing the class  $\alpha$ .*

Note, by purity for the Brauer group [Gro68, Thm. III.6.1], we have  $\text{Br}(\widehat{X}) = \text{Br}(X)$ , so  $\alpha$  is uniquely determined by the cocycle  $\widehat{\beta}$ .

*Proof.* Take  $V \subset \widehat{Y}$  nonempty open such that  $\psi_0 = \pi|_V$  is a finite étale covering of some open subscheme of  $\widehat{X}$ . Let  $\psi: \widehat{W} \rightarrow \widehat{X}$  be the normalization of  $\widehat{X}$  in  $(\psi_0)_* \mathcal{O}_V$ . Shrinking  $\widehat{X}$  (and maintaining that its complement in  $X$  has codimension 2) we may suppose that  $\widehat{W}$  is smooth. By the universal property of the normalization ([EGAII,

6.3.9]) there is a (unique) lift  $\widehat{Y} \rightarrow \widehat{W}$  of  $\pi$ . Consider the element of  $Z^2(\widehat{Y} \times_{\widehat{X}} \widehat{W} \rightarrow \widehat{W}, \mathbb{G}_m)$  obtained by restricting  $\widehat{\beta}$ . The further restriction to  $Z^2(\widehat{Y} \times_{\widehat{X}} \widehat{Y} \rightarrow \widehat{Y}, \mathbb{G}_m)$  is (explicitly) a coboundary, we apply Lemma 5 and observe that by the proof, from the fact that  $\widehat{W} \rightarrow \widehat{X}$  is finite and hence universally closed, the Zariski refinement may be taken to come from a Zariski refinement of  $\widehat{Y}$ , i.e., we obtain  $\widehat{\gamma} \in C^1(\coprod \widehat{Y}_i \times_{\widehat{X}} \widehat{W} \rightarrow \widehat{W}, \mathbb{G}_m)$  whose coboundary is the restriction of  $\widehat{\beta}$ . Using the flatness of  $\widehat{W} \rightarrow \widehat{X}$ , we may regard  $\widehat{\gamma}$  as patching data for a sheaf of Azumaya algebras over  $\widehat{X}$  as in [Mil80, Prop. IV.2.11], whose class in the Brauer group is that of  $\widehat{\beta}$ . Pushforward via  $\widehat{X} \rightarrow X$  may be computed by making an arbitrary extension as a coherent sheaf, and forming the double dual. This is then a sheaf of Azumaya algebras on  $X$  by [Gro68, Thm. I.5.1(ii)].  $\square$

#### 4. PROOF OF THEOREM 1

The proof of Theorem 1 is carried out in several steps.

*Step 1.* (Proposition 7) We obtain a nonempty open subscheme  $X^\circ$  of  $X$ , a finite Galois extension  $K$  of  $k$ , and a sequence of elements

$$(\alpha_1, \dots, \alpha_N) \subset \text{Br}(X_K^\circ)$$

for some  $N$  which generate a subgroup of  $\text{Br}(X_K^\circ)$  containing  $\text{Br}(X_{\bar{k}})[n]$ . We obtain an étale covering  $Y^\circ \rightarrow X^\circ$ , such that each  $\alpha_i$  is given by an explicit 2-cocycle for the étale cover  $Y_K^\circ \rightarrow X_K^\circ$ .

*Step 2.* (Proposition 9) Given  $\alpha \in \text{Br}(X_K^\circ)$  defined by an explicit cocycle, we provide an effective procedure to test whether  $\alpha$  vanishes in  $\text{Br}(X_{\bar{k}})$ , and in case of vanishing, to produce a 1-cochain lift of the cocycle, defined over some effective extension of  $K$ . We use this procedure in two ways.

- (i) By repeating Step 1 with another open subscheme  $\widetilde{X}^\circ$ , with  $X \setminus (X^\circ \cup \widetilde{X}^\circ)$  of codimension 2 (or empty), to identify the geometrically *unramified* Brauer group elements, i.e., those in the image of  $\text{Br}(X_K) \rightarrow \text{Br}(X_K^\circ)$  after possibly extending  $K$ .
- (ii) To identify those  $\alpha$  such that  $\alpha$  and  ${}^g\alpha$  have the same image in  $\text{Br}(X_{\bar{k}})$  for all  $g \in \text{Gal}(K/k)$ . Again after possibly extending  $K$  (remaining finite Galois over  $k$ ), we may suppose that all such  $\alpha$  satisfy  $\alpha = {}^g\alpha$  in  $\text{Br}(X_K)$  for all  $g \in \text{Gal}(K/k)$ .

The result is a sequence of elements

$$(\alpha'_1, \dots, \alpha'_M) \subset \text{Br}(X_K)[n]^{\text{Gal}(K/k)},$$

each given by a cocycle over  $X_K^\circ$  as well as one over  $\widetilde{X}_K^\circ$ , generating  $\text{Br}(X_{\bar{k}})[n]^{\text{Gal}(\bar{k}/k)}$ .

*Step 3.* (Proposition 10) Combine the data from the Galois invariance of the  $\alpha'_i$  and the alternate representation over  $\widetilde{X}_K^\circ$  to obtain cocycle representatives of each  $\alpha'_i$  defined over the complement of a codimension 2 subset of  $X$ , as well as cochains there that encode the Galois invariance.

*Step 4.* (Proposition 11) For every Galois-invariant  $n$ -torsion element of  $\text{Br}(X_{\bar{k}})$ , with representing cocycle defined over  $K$  obtained in Step 3, compute the obstruction

to the existence of an element of  $\mathrm{Br}(X)$  having the same image class in  $\mathrm{Br}(X_{\bar{k}})$ . When the obstruction vanishes, produce a cocycle representative of such an element of  $\mathrm{Br}(X)$ , defined over the complement of a codimension 2 subset of  $X$ . Each such element of  $\mathrm{Br}(X)$  will be unique up to an element of  $\ker(\mathrm{Br}(X) \rightarrow \mathrm{Br}(X_K))$ , the algebraic part of the Brauer group, which has been treated in [KT08].

*Step 5.* From the cocycle representatives of elements of  $\mathrm{Br}(X)$  obtained in Step 4, produce sheaves of Azumaya algebras defined globally on  $X$  (Lemma 6).

*Step 6.* Compute local invariants. A sheaf of Azumaya algebras may be effectively converted to a collection of representing 2-cocycles, each for a finite étale covering of some  $U_i$  with  $(U_i)$  a Zariski covering of  $X$  ([Gro68, Thm. I.5.1(iii), 5.10]). Then we are reduced to the local analysis described in [KT08, §9].

## 5. GENERATORS OF $\mathrm{Br}(X_{\bar{k}})[n]$ BY FIBRATIONS

For the first step, we produce generators of the  $n$ -torsion in the Brauer group of  $\bar{X} := X_{\bar{k}}$ . Starting from  $X \subset \mathbb{P}^N$ , a general projection to  $\mathbb{P}^1$  yields, after replacing  $X$  by its blow-up at finitely many points, a fibration

$$f: X \rightarrow \mathbb{P}^1 \quad (5.1)$$

with geometrically connected fibers. By removing the exceptional divisors from the codimension 2 complement in Step 5 and viewing it as a codimension 2 complement of  $X$ , the proof of Theorem 1 is reduced to the case that  $f$  as in (5.1) exists.

Notice that, given a finite set of divisors on  $X$ , (5.1) may be chosen so that each of these divisors maps dominantly to  $\mathbb{P}^1$ .

That the  $n$ -torsion in the Brauer group of a smooth projective surface over  $\bar{k}$  may be computed using a fibration is standard. We include a sketch of a proof, for completeness.

**Proposition 7.** *Let  $X$  be a smooth projective geometrically irreducible surface over a number field  $k$ , and let  $f: X \rightarrow \mathbb{P}^1$  be a nonconstant morphism with connected geometric fibers, both given by explicit equations. Let  $n$  be a given positive integer. Then there exist, effectively:*

- (i) a finite Galois extension  $K$  of  $k$ ,
- (ii) a nonempty open subset  $S \subset \mathbb{P}^1$ ,
- (iii) an étale covering  $S' \rightarrow S$ ,
- (iv) 2-cocycles of rational functions for the covering  $X_K \times_{\mathbb{P}_K^1} S'_K \rightarrow X_K \times_{\mathbb{P}_K^1} S_K$ ,

such that  $\mathrm{Br}(X_{\bar{k}} \times_{\mathbb{P}_{\bar{k}}^1} S_{\bar{k}})[n]$  is spanned by the classes of the 2-cocycles, base-extended to  $\bar{k}$ .

*Proof.* We let  $S \subset \mathbb{P}^1$  denote the maximal subset over which  $f$  is smooth, and  $X^\circ = f^{-1}(S)$ . By the exact sequences of Section 2, it suffices to carry out following tasks (perhaps for a larger value of  $n$ ):

- (1) Compute  $H^1(S_{\bar{k}}, \mathrm{Pic}_{X_{\bar{k}}^\circ/S_{\bar{k}}}[n])$  by means of cocycles.
- (2) Find divisors on  $X_{\bar{k}}$  whose classes in  $\mathrm{Pic}(X_{\bar{k}}^\circ/S_{\bar{k}})$  represent the elements appearing in these cocycles.
- (3) Find explicit 2-cocycle representatives of elements of  $\mathrm{Br}(X_{\bar{k}}^\circ)$  which correspond to these elements by the isomorphism (2.1).

The field  $K$  is an explicit suitable extension, over which the steps are carried out. Step (1) is clear, since there is an explicit finite étale covering  $C \rightarrow S$  trivializing  $\text{Pic}_{X^\circ/S}[n]$ . Then there is a finite étale covering  $S' \rightarrow C$ , with  $S'_k \rightarrow C_k$  a product of cyclic étale degree  $n$  covers, such that  $S'_k \rightarrow S_k$  trivializes  $H^1(S_k, \text{Pic}_{X_k^\circ/S_k}[n])$ . (The proof of Lemma 2 provides an effective procedure to compute  $S'$ , using effective Jacobian arithmetic.) Step (2) can be carried out effectively as described in [KT08, §4], using an effective version of Tsen's theorem (for the function field, this is standard, see e.g. [Pr], then apply Lemma 5). On the level of cocycles, the Leray spectral sequence (2.1) gives rise to a 3-cocycle, and Step (3) can be carried out as soon as this is represented as a coboundary, which we have again possibly after making a Zariski refinement of  $S'$  (cf. [Mil80, Exa. III.2.22(d)]). An explicit description of the procedure to produce the 3-cocycle using the Leray spectral sequence may be found in [KT08, Prop. 6.1].  $\square$

## 6. RELATIONS AMONG GENERATORS

In this section we show how to compare elements of the Brauer group of a Zariski open subset of a smooth projective surface  $\overline{X}$  over  $\bar{k}$ , under the assumption that the geometric Picard group  $\text{Pic}(\overline{X})$  is finitely generated, and  $\overline{X}$  as well as a finite set of divisors generating  $\text{Pic}(\overline{X})$  are explicitly given. The method goes back to Brauer [Bra28], with refinements in [Bra32].

**Lemma 8.** *Let  $X^\circ$  be a smooth quasi-projective geometrically irreducible surface over a number field  $k$ ,  $Z^\circ \rightarrow X^\circ$  a finite étale morphism, and  $\beta \in Z^2(Z^\circ \rightarrow X^\circ, \mathbb{G}_m)$  a Čech cocycle representative of an element  $\alpha \in \text{Br}(X^\circ)$ . We suppose  $X^\circ$ ,  $Z^\circ$  and  $\beta$  are given by explicit equations, respectively functions. Let  $n$  be a given positive integer, and  $\gamma \in C^1(Z^\circ \rightarrow X^\circ, \mathbb{G}_m)$  a Čech cochain whose coboundary is equal to  $n \cdot \beta$ . We suppose that  $k$  contains the  $n$ -roots of unity, and that an identification  $\mu_n \simeq \mathbb{Z}/n\mathbb{Z}$  is fixed. Then there exists, effectively, a finite group  $G$ , a finite étale morphism  $Y^\circ \rightarrow Z^\circ$ , a  $G$ -torsor structure on  $Y^\circ \rightarrow X^\circ$ , a central extension of finite groups*

$$1 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow H \rightarrow G \rightarrow 1, \quad (6.1)$$

*and a 1-cochain  $\delta \in C^1(Y^\circ \rightarrow X^\circ, \mathbb{G}_m)$  such that image in  $\text{Br}(X^\circ) = H^2(X^\circ, \mathbb{G}_m)$  of the induced element of  $H^2(X^\circ, \mathbb{Z}/n\mathbb{Z}) \simeq H^2(X^\circ, \mu_n)$  is equal to  $\alpha$ , and the coboundary  $\delta$  is the difference between the latter and the given  $\beta$  refined by  $Y^\circ \rightarrow Z^\circ$ .*

*Proof.* There exists a finite étale cover  $Y^\circ$  of  $Z^\circ$ , such that the restriction of  $\gamma$  to  $Z^\circ \times_{X^\circ} Z^\circ$  is an  $n$ th power. It follows that the restriction of  $\beta$  differs by a coboundary from an element of the image of  $Z^2(Y^\circ \rightarrow X^\circ, \mu_n)$ . These can be produced explicitly. Upon further refinement of  $Y^\circ$ , we may suppose that  $Y^\circ$  is irreducible,  $Y^\circ \rightarrow X^\circ$  is a Galois  $G$ -covering for some finite group  $G$ , and then the cocycle condition is precisely the condition to be a 2-cocycle for the group cohomology of  $G$  with values in  $\mathbb{Z}/n\mathbb{Z}$  (with trivial  $G$ -action on  $\mathbb{Z}/n\mathbb{Z}$ ). This gives us (6.1).  $\square$

**Proposition 9.** *Let  $X$  be a smooth projective geometrically irreducible surface over a number field  $k$  with finitely generated geometric Picard group  $\text{Pic}(X_{\bar{k}})$ . Let  $X^\circ$  be a nonempty open subscheme,  $\pi: Y^\circ \rightarrow X^\circ$  an étale cover, and  $\beta \in Z^2(Y^\circ \rightarrow X^\circ, \mathbb{G}_m)$  a Čech cocycle representative of an element  $\alpha \in \text{Br}(X^\circ)$ . Let  $n$  be a given positive integer, and  $\gamma \in C^1(Y^\circ \rightarrow X^\circ, \mathbb{G}_m)$  a Čech cochain whose coboundary is equal to  $n \cdot \beta$ . We suppose  $X$ , a finite set of divisors generating  $\text{Pic}(X_{\bar{k}})$ ,  $X^\circ$ ,  $Y^\circ$ ,  $\pi$ ,  $\beta$ ,*



and  $\gamma$  are given by explicit equations, respectively functions. Then there exists an effective procedure to determine whether  $\alpha_{\bar{k}} = 0$  in  $\text{Br}(X_{\bar{k}}^\circ)$ , and in case  $\alpha_{\bar{k}} = 0$ , to produce a finite extension  $K$  of  $k$ , a Zariski open covering  $(Y_i^\circ)$  of  $Y^\circ$ , and a 1-cochain  $\delta \in C^1(\coprod (Y_i^\circ)_K \rightarrow X_K^\circ, \mathbb{G}_m)$ , whose coboundary is equal to the base-extension to  $K$  of the refinement of  $\beta$  by  $\coprod Y_i^\circ \rightarrow Y^\circ$ .

*Proof.* It suffices to prove the result after an effective shrinking of  $X^\circ$  and extension of the base field, by Lemma 5 (we note that the Zariski open subsets that are produced in the proof may be taken to be Galois invariant) and, by Lemma 4, after a refinement of the given cover. So we may suppose that  $\pi$  is finite,  $\text{Pic}(X_k^\circ) = 0$ , the field  $k$  contains the  $n$ th roots of unity (with a fixed identification  $\mathbb{Z}/n\mathbb{Z} \simeq \mu_n$ ), and the cocycle  $\beta$  takes its values in  $\mu_n$  (Lemma 8) and is the universal one for a  $G$ -torsor structure on  $Y^\circ \rightarrow X^\circ$  and an extension (6.1). Without loss of generality,  $Y^\circ$  is geometrically irreducible, and the class of the extension in  $H^2(G, \mathbb{Z}/n\mathbb{Z})$  (group cohomology for  $\mathbb{Z}/n\mathbb{Z}$  with trivial  $G$ -action) is not annihilated by any positive integer smaller than  $n$ . It follows from  $\text{Pic}(X_k^\circ) = 0$  that  $\alpha_{\bar{k}} = 0$  in  $\text{Br}(X_{\bar{k}}^\circ)$  if and only if the class of  $\beta$  is 0 in  $H^2(X_{\bar{k}}^\circ, \mu_n)$ .

The Leray spectral sequence gives rise to an exact sequence

$$0 \rightarrow H^1(G, \mu_n) \rightarrow H^1(X_{\bar{k}}^\circ, \mu_n) \rightarrow H^1(Y_{\bar{k}}^\circ, \mu_n)^G \rightarrow H^2(G, \mu_n) \rightarrow H^2(X_{\bar{k}}^\circ, \mu_n).$$

It follows that the class of  $\beta$  is 0 in  $\text{Br}(X_{\bar{k}}^\circ)$  if and only if there exists an irreducible finite étale covering  $\bar{Y}^\circ$  of  $\bar{Y}^\circ := Y_{\bar{k}}^\circ$ , cyclic of degree  $n$ , admitting a structure of  $H$ -torsor over  $\bar{X}^\circ := X_{\bar{k}}^\circ$  compatible with the  $G$ -torsor structure on  $\bar{Y}^\circ$ . This can be tested, provided that we can explicitly generate all degree  $n$  cyclic étale coverings of  $\bar{Y}^\circ$ . If we have such a covering, we take  $K$  so that the covering and  $H$ -torsor structure are defined over  $K$ , then the restriction of  $\beta$  to the covering is explicitly a coboundary.

Choose an explicit fibration  $\tau: \bar{Y}^\circ \rightarrow \mathbb{P}^1$ . Since we may shrink  $\bar{Y}^\circ$ , we may replace  $\bar{Y}^\circ$  by the preimage of Zariski open  $T \subsetneq \mathbb{P}^1$ , chosen so that the geometric fibers are complements of exactly some number  $\ell$  of distinct points in a smooth irreducible curve of some genus  $g$ , these  $\ell$  points being the fibers of a finite étale cover of  $T$ .

By the Leray spectral sequence, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{O}(T)^*/(\mathcal{O}(T)^*)^n & \longrightarrow & H^1(\bar{Y}^\circ, \mu_n) & \longrightarrow & H^0(T, R^1\tau_*\mu_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow \bar{k}(T)^*/(\bar{k}(T)^*)^n & \longrightarrow & H^1(\bar{Y}_{\bar{k}(T)}^\circ, \mu_n) & \longrightarrow & H^0(\bar{k}(T), R^1(\tau_{\bar{k}(T)})_*\mu_n) & \longrightarrow & 0 \end{array}$$

where we have used  $H^2(T, \mu_n) = H^2(\bar{k}(T), \mu_n) = 0$ . Arguing as in [Mil80, proof of Lemma III.3.15] we see that  $R^1\tau_*\mu_n$  is a locally constant torsion sheaf with finite fibers. It follows that the right-hand vertical map is an isomorphism. By the snake lemma, the leftmost two vertical maps are injective and have isomorphic cokernels.

Let  $\bar{Y}_{\bar{k}(T)}$  be a smooth projective curve containing  $\bar{Y}_{\bar{k}(T)}^\circ$  as an open subscheme. We can find generators of  $H^1(\bar{Y}_{\bar{k}(T)}^\circ, \mu_n)$ , modulo  $H^1(\bar{k}(T), \mu_n) = \bar{k}(T)^*/(\bar{k}(T)^*)^n$  by computing the  $\bar{k}(T)$ -rational  $n$ -torsion points of the Jacobian and (using effective Tsen's theorem) lifting these to divisor representatives. Lemma 2 supplies additional generators of  $H^1(\bar{Y}_{\bar{k}(T)}^\circ, \mu_n)$ : for elements of  $\bigoplus \mathbb{Z}/n\mathbb{Z}$  (sum over points of  $\bar{Y}_{\bar{k}(T)} \setminus \bar{Y}_{\bar{k}(T)}^\circ$ ) of weighted (by degree) sum 0, we test whether a fiber of a multiplication by  $n$  map of Jacobians has a  $\bar{k}(T)$ -rational point (and again use effective Tsen's

theorem to produce divisor representatives). Each generator of  $H^1(\overline{Y}_{\bar{k}(T)}^\circ, \mu_n)$ , modulo  $H^1(\bar{k}(T), \mu_n)$ , may be effectively lifted to  $H^1(\overline{Y}^\circ, \mu_n)$  by a diagram chase, using the isomorphism of the cokernels of left two vertical morphisms in the diagram.  $\square$

## 7. GALOIS INVARIANTS IN $\text{Br}(X_{\bar{k}})[n]$

In this section, we focus on the problem of deciding whether a Galois invariant element in  $\text{Br}(X_{\bar{k}})[n]$  lies in the image in  $\text{Br}(X_{\bar{k}})$  of an element of  $\text{Br}(X_K)^{\text{Gal}(K/k)}$ . Concretely, this amounts to adjusting elements of  $\text{Br}(X_K)$  by elements of  $\text{Br}(K)$ , when possible, so that they become  $\text{Gal}(K/k)$ -invariant. This will be done effectively.

**Proposition 10.** *Let  $X$  be a smooth projective geometrically irreducible surface over a number field  $k$ . Let  $K$  be a finite Galois extension of  $k$ . Let  $X^\circ$  and  $\tilde{X}^\circ$  be open subschemes whose union is the complement of a subset that has codimension 2 (or is empty),  $Y^\circ \rightarrow X^\circ$  and  $\tilde{Y}^\circ \rightarrow \tilde{X}^\circ$  étale coverings,  $\beta \in Z^2(Y_K^\circ \rightarrow X_K^\circ, \mathbb{G}_m)$  and  $\tilde{\beta} \in Z^2(\tilde{Y}_K^\circ \rightarrow \tilde{X}_K^\circ, \mathbb{G}_m)$  cocycles, and  $\delta_g \in C^1(Y_K^\circ \rightarrow X_K^\circ, \mathbb{G}_m)$  having coboundary  $\beta - {}^g\beta$  for every  $g \in \text{Gal}(K/k)$ . Assume that  $\beta$  and  $\tilde{\beta}$  give rise to the same class in  $\text{Br}((X^\circ \cap \tilde{X}^\circ)_{\bar{k}})$ . Then we may effectively produce an open subscheme  $\hat{X} \subset X$ , containing  $X^\circ$ , whose complement has codimension 2 (or is empty), an étale cover  $\hat{Y} \rightarrow \hat{X}$ , a finite extension  $L$  of  $K$ , Galois over  $k$ , a cocycle  $\hat{\beta} \in Z^2(\hat{Y} \rightarrow \hat{X}, \mathbb{G}_m)$  giving rise to the same class as  $\beta$  in  $\text{Br}(X_K^\circ)$ , and cochain  $\hat{\delta}_g \in C^1(\hat{Y} \rightarrow \hat{X}, \mathbb{G}_m)$  having coboundary  $\hat{\beta} - {}^g\hat{\beta}$ , for all  $g \in \text{Gal}(L/k)$ .*

*Proof.* Let  $\xi_1, \dots, \xi_N$  denote the codimension 1 generic points of  $Y^\circ \times_X \tilde{Y}^\circ$  whose image in  $\tilde{X}^\circ$  is one of the generic points in  $X$  of the codimension 1 irreducible components of  $X \setminus X^\circ$ . We may apply Proposition 9 to the covering  $Y^\circ \times_X \tilde{Y}^\circ \rightarrow X^\circ \cap \tilde{X}^\circ$  and insist that one of the open sets that is produced, in addition to being Galois invariant, contains all the points above  $\xi_1, \dots, \xi_N$ . (The field that emerges, enlarged if necessary, is taken as the field  $L$  mentioned in the statement.) Call the open set  $U$ . We replace  $\tilde{X}^\circ$  with the complement of the closure of the image of the complement of  $U$  in  $Y_L^\circ \times_{X_L} \tilde{Y}_L^\circ$ , and restrict  $\tilde{Y}^\circ$  accordingly. Now we have  $U = Y_L^\circ \times_{X_L} \tilde{Y}_L^\circ$ , so we may apply Lemma 3 to produce  $\hat{\beta} \in Z^2(Y_L^\circ \amalg \tilde{Y}_L^\circ \rightarrow X_L^\circ \cup \tilde{X}_L^\circ, \mathbb{G}_m)$ . We apply Lemma 5 to produce  $\hat{\delta}_g$ , which involves replacing  $Y^\circ$  and  $\tilde{Y}^\circ$  by Zariski covers.  $\square$

**Proposition 11.** *Let  $X$  be a smooth projective geometrically irreducible variety over a number field  $k$ , given by explicit equations, let  $K$  be a finite Galois extension of  $k$ , and assume that  $\text{Pic}(X_{\bar{k}})$  is torsion-free, generated by finitely many explicitly given divisors, defined over  $K$ . Let  $\alpha \in \text{Br}(X_{\bar{k}})$  be given by means of a cocycle representative  $\beta \in Z^2(\hat{Y}_K \rightarrow \hat{X}_K, \mathbb{G}_m)$ , where  $\hat{Y}_K \rightarrow \hat{X}_K$  is an étale cover, with  $\hat{X}$  an open subscheme of  $X$  whose complement has codimension at least 2 (or is empty). Assume given  $\delta^{(g)} \in C^1(\hat{Y}_K \rightarrow \hat{X}_K, \mathbb{G}_m)$ , having coboundary  $\beta - {}^g\beta$ , for every  $g \in \text{Gal}(K/k)$ . Then there exists an effective computable obstruction in  $H^2(\text{Gal}(K/k), \text{Pic}(X_K))$  to the existence of  $\alpha_0 \in \text{Br}(X)$  such that  $\alpha_0$  and  $\alpha$  have the same image in  $\text{Br}(X_{\bar{k}})$ . When the obstruction class vanishes, we can effectively construct a cocycle representative of  $\alpha_0|_{\hat{X}}$  in  $Z^2(\hat{Y}_K \rightarrow \hat{X}, \mathbb{G}_m)$  for some  $\alpha_0 \in \text{Br}(X)$  satisfying  $(\alpha_0)_K = \alpha$ .*

*Proof.* By the Leray spectral sequence, we have an exact sequence

$$\text{Br}(X) \rightarrow \ker \left( \text{Br}(X_K)^{\text{Gal}(K/k)} \rightarrow H^2(\text{Gal}(K/k), \text{Pic}(X_K)) \right) \rightarrow H^3(\text{Gal}(K/k), K^*).$$

Also note that the nontriviality in  $H^2(\text{Gal}(K/k), \text{Pic}(X_K))$  implies the nontriviality in  $H^2(\text{Gal}(L/k), \text{Pic}(X_L))$  for any finite extension  $L$  of  $K$ , Galois over  $k$ , by the Hochschild-Serre spectral sequence

$$0 = H^1(\text{Gal}(L/K), \text{Pic}(X_L))^{\text{Gal}(K/k)} \rightarrow H^2(\text{Gal}(K/k), \text{Pic}(X_K)) \rightarrow H^2(\text{Gal}(L/k), \text{Pic}(X_L)).$$

The hypothesis concerning  $\delta^{(g)}$  may be written

$$\frac{\delta^{(g)}(y, y')\delta^{(g)}(y', y'')}{\delta^{(g)}(y, y'')} = \frac{\beta(y, y', y'')}{g\beta(y, y', y'')} \quad (7.1)$$

and implies that

$$\frac{\delta^{(g)} g\delta^{(g')}}{\delta^{(gg')}} \in Z^1(\widehat{Y}_K \rightarrow \widehat{X}_K, \mathbb{G}_m) \quad (7.2)$$

for every  $g, g' \in \text{Gal}(K/k)$ . Arguments as in [KT08, §6] show that (7.2) gives the obstruction class in  $H^2(\text{Gal}(K/k), \text{Pic}(X_K))$ . Of course, each cocycle (7.2) may be explicitly represented by a divisor, whose class in  $\text{Pic}(X_K)$  is then readily computed.

Assuming that the obstruction class in  $H^2(\text{Gal}(K/k), \text{Pic}(X_K))$  vanishes, each  $\delta^{(g)}$  may be modified by a cocycle so that each element (7.2) is a coboundary, i.e., so that there exist  $\varepsilon^{(g, g')} \in \mathcal{O}_{\widehat{Y}_K}^*$  satisfying

$$\frac{\varepsilon^{(g, g')}(y')}{\varepsilon^{(g, g')}(y)} = \frac{\delta^{(g)}(y, y')g\delta^{(g')}(y, y')}{\delta^{(gg')}(y, y')}. \quad (7.3)$$

In this case the divisor representative of (7.2) is a principal divisor, hence the divisor associated to an effectively computable rational function.

Combining (7.1) and (7.3), we have

$$\frac{\varepsilon^{(g, g')}(y)\varepsilon^{(gg', g'')}(y)}{\varepsilon^{(g, g'g'')}(y)g\varepsilon^{(g', g'')}(y)} = \frac{\varepsilon^{(g, g')}(y')\varepsilon^{(gg', g'')}(y')}{\varepsilon^{(g, g'g'')}(y')g\varepsilon^{(g', g'')}(y')},$$

hence

$$\varepsilon^{(g, g')}\varepsilon^{(gg', g'')}/(\varepsilon^{(g, g'g'')}g\varepsilon^{(g', g'')}) \in \mathcal{O}_{\widehat{X}_K}^*, \quad (7.4)$$

i.e., is a constant function, for every  $g, g', g'' \in \text{Gal}(K/k)$ . The rest of the argument is similar to [KT08, Prop. 6.3]. The constants (7.4) determine a class in  $H^3(\text{Gal}(K/k), K^*)$ , which may be effectively tested for vanishing. In case of nonvanishing a further finite extension may be effectively computed, which kills this class. In case of vanishing, a 2-cochain lift is effectively produced. Modifying  $\varepsilon^{(g, g')}$ , then, yields

$$\varepsilon^{(g, g')}(y)\varepsilon^{(g', g'')}(y) = \varepsilon^{(g, g'g'')}(y)g\varepsilon^{(g', g'')}(y) \quad (7.5)$$

Now if we set

$$\beta^{(g, g')}(y, y', y'') = \frac{\beta(y, y', y'')\varepsilon^{(g, g')}(y'')}{\delta^{(g)}(y', y'')}$$

then we have

$$\beta^{(g, g')}(y, y', y'')\beta^{(gg', g'')}(y, y'', y''') = \beta^{(g, g'g'')}(y, y', y''')g\beta^{(g', g'')}(y', y'', y'''),$$

i.e., we have an element of  $Z^2(\widehat{Y}_K \rightarrow \widehat{X}, \mathbb{G}_m)$  determining an element  $\alpha_0 \in H^2(X, \mathbb{G}_m)$ . The restriction to  $\widehat{X}_K$  is defined by the cocycle  $\beta^{(e, e)}$ , which is equal to  $\beta$ , up to coboundary.  $\square$

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